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WHY THE Σ AND Ξ HYPERFINE SPLITTINGS ARE DIFFERENT*

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ABSTRACT

The Σ and Ξ splittings, predicted to be equal in the SU(6) quark model, are shown to differ because the smaller size of the Ξ wave function enhances the short range hyperfine interaction. The change in the relative motion of a u-s pair in the hyperon produced by a change in the mass of the third quark is not a simple scale change and is sensitive to details of the interquark potential. This effect is absent in the harmonic oscillator model, where the third quark is completely decoupled from the relative motion within the other pair, but is appreciable and has the right order of magnitude in the logarithmic potential model.

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The mass differences between corresponding baryon states in the $1/2^+$ octet and the $3/2^+$ decuplet have been described in the quark model as due to a two-body hyperfine interaction between quark pairs.[1,2,3] The relations between hyperfine splittings in different baryons have been found to be in good agreement with experiment. But the discrepancy with the prediction

$$M(\Xi^*) - M(\Xi) = M(\Sigma^*) - M(\Sigma) \tag{1}$$

is puzzling.[4] The LHS is 216 MeV, the RHS is 192 MeV. Although this difference is only 12% and could be dismissed as less than the precision expected for such a crude model, it is tempting to ask whether this difference is a physically interesting signal above the noise.

The pair of identical quarks, nonstrange in the Σ, strange in the Ξ, is always in the spin triplet state and does not contribute to the hyperfine splitting. With SU(3) symmetric baryon wave functions the strange-nonstrange pairs whose spin couplings are different in the $3/2^+$ and $1/2^+$ states have the same wave functions in the Σ and Ξ and therefore give the same hyperfine splitting.

A discrepancy with the relation (1) suggests that SU(3) breaking makes the wave functions of the strange-nonstrange pairs different in the Σ and the Ξ. Since the Ξ is the more massive, it is natural to suggest that the Ξ wave function has a smaller radius.[4] The short range hyperfine interaction is then larger. However, quantitative investigation[5] of SU(3) symmetry breaking in the Isgur-Karl harmonic oscillator model[6] does not significantly alter the prediction (1).

We point out in this paper that the null result in the Isgur-Karl model arises from peculiarities of the harmonic oscillator potential, and that effects of the right sign and order of magnitude are obtained with other potentials, such as the Quigg-Rosner logarithmic potential.[7] The underlying physical difference is the separability of the two relative degrees of freedom found only in the harmonic potential and not in other potentials. In the harmonic oscillator model the relative motion of one quark with respect to the center of mass of the other pair is completely decoupled from the relative motion within the pair. Changing the mass of the odd quark only changes the size of its motion relative to the other pair. The size of the pair wave function is unaffected. In other potentials the motion of the third quark is coupled to the overall size of the entire wave function.

This effect could be calculated quantitatively by solving the three-body problem for the case of unequal masses in different potentials. Rather than undertaking such an ambitious program we obtain approximate estimates of the effect by simpler methods. Qualitative and semi-quantitative features are obtained by using the virial theorem and scaling properties of wave functions for systems with nearly equal masses, with a small mass difference considered as a perturbation.

Consider a simplified model in which the strange-nonstrange pair whose hyperfine interaction is being studied is replaced by a pair of identical quarks whose reduced mass is equal to the mass of the strange-nonstrange pair. Our fictitious quarks then have the mass

$$m_q = 2m_u m_s/(m_u + m_s) \ . \tag{2}$$

We study the change in the hyperfine interaction of this quark pair, denoted as quarks number 1 and 2, as a function of the mass m_3 of the third quark. The qualitative features of this effect can be seen by examining the behavior of the kinetic energy of the relative motion of the 12 pair,

$$T_{12} = p_{12}^2/2m_{12} \tag{3a}$$

where $\vec{p}_{ij} = (\vec{p}_i - \vec{p}_j)/2$ is the relative momentum of the pair ij and m_{ij} is the reduced mass of the pair. Since only m_3 is changed, $m_{31} = m_{23}$ is changed but m_{12} remains unchanged. The expectation value of the relative kinetic energy is given by the virial theorem

$$\begin{aligned} \langle T_{12} \rangle = \langle \frac{1}{2} r_{12} \frac{\partial V}{\partial r_{12}} \rangle = \frac{1}{2} \langle r_{12} \frac{dV_{12}}{dr_{12}} + r_{12} \left(\frac{dV_{23}}{dr_{23}}\right) \frac{(r_{12}^2 + r_{23}^2 - r_{31}^2)}{4r_{23}r_{12}} \\ + r_{12}\left(\frac{dV_{31}}{dr_{31}}\right) \frac{(r_{12}^2 + r_{31}^2 - r_{23}^2)}{4r_{31}r_{12}} \rangle , \end{aligned} \tag{3b}$$

where the potential V is assumed to be the sum of two-body potentials

$$V = V_{12}(r_{12}) + V_{23}(r_{23}) + V_{31}(r_{31}) . \tag{3c}$$

The hyperfine interaction in a two-body system is assumed to be proportional to the square of the wave function at the origin. The expectation value of this interaction is related by a well known theorem to the expectation value of the derivative of the potential.[8] This theorem gives an expression very similar to (3b) for the three-body system,

$$V_{12}^{hyp} = Km_{12} \langle \frac{dV_{12}}{dr_{12}} + \left(\frac{dV_{23}}{dr_{23}}\right) \frac{(r_{12}^2 + r_{23}^2 - r_{31}^2)}{4r_{23}r_{12}} + \left(\frac{dV_{31}}{dr_{31}}\right) \frac{(r_{12}^2 + r_{31}^2 - r_{23}^2)}{4r_{31}r_{12}} \rangle \tag{3d}$$

where K is a constant depending upon the strength of the interaction and the particle masses.

Consider the case of a power law potential

$$\frac{dV}{dr} = Ur^{\alpha-1} \tag{4}$$

where the derivative has been used in the definition (4) in order to include the case of the log potential, $\alpha = 0$, as well as all power law potentials. Substituting Eq.(4) into Eq.(3b) then gives

$$\begin{aligned} \langle T_{12} \rangle &= \frac{U}{2} \langle r_{12}^{\alpha} + \frac{1}{4} (r_{12}^2 r_{23}^{\alpha-2} + r_{23}^{\alpha} - r_{31}^2 r_{23}^{\alpha-2} + r_{12}^2 r_{31}^{\alpha-2} + r_{31}^{\alpha} - r_{23}^2 r_{31}^{\alpha-2}) \rangle \\ &= \frac{U}{2} \langle r_{12}^{\alpha} + \frac{1}{4} (r_{23}^{\alpha} + r_{31}^{\alpha}) + \frac{1}{4} [r_{12}^2 (r_{23}^{\alpha-2} + r_{31}^{\alpha-2}) - (r_{31}^2 r_{23}^{\alpha-2} + r_{23}^2 r_{31}^{\alpha-2}) \rangle \ . \end{aligned} \tag{5a}$$

This can be rewritten

$$\langle T_{12} \rangle = \frac{U}{2} \langle r_{12}^{\alpha} [1 + \frac{1}{4} \{(r_{12}/r_{23})^{2-\alpha} + (r_{12}/r_{31})^{2-\alpha} + (r_{23}^{\alpha-2} - r_{31}^{\alpha-2})(r_{23}^2 - r_{31}^2) r_{12}^{-\alpha}\}] \rangle \ . \tag{5b}$$

Similarly

$$V_{12}^{hyp} = Km_{12}\langle r_{12}^{\alpha-1}[1 + \frac{1}{4}\{(r_{12}/r_{23})^{2-\alpha} + (r_{12}/r_{31})^{2-\alpha}$$

$$+ (r_{23}^{\alpha-2} - r_{31}^{\alpha-2})(r_{23}^{2} - r_{31}^{2})r_{12}^{-\alpha}\}]\rangle \ . \tag{5c}$$

These expressions (5) illustrate the qualitative behavior of the relative motion of the 12 pair as a function of the mass m_3. For the symmetric case, $m_1 = m_2 = m_3$, expectation values are symmetric under any permutations of the indices 1, 2 and 3. The two terms in the square bracket of Eq.(5a) cancel and

$$\langle T_{12}\rangle_{sym} = \langle T_{23}\rangle_{sym} = \langle T_{31}\rangle_{sym} = \frac{3U}{4}\langle r_{ij}^{\alpha}\rangle \ . \tag{6a}$$

For the harmonic oscillator potential, $\alpha = 2$, Eq.(5a) simplifies and the result (6a) is seen to hold for all values of m_3. Thus $\langle T_{12}\rangle$ is independent of the motion and mass of particle 3, as expected. However, for $\alpha < 2$, the expression depends upon the motion of particle 3. In particular, in the limit $m_3 \to 0$, where $r_{23} \approx r_{31} \gg r_{12}$, $\frac{r_{12}}{r_{23}} \sim \frac{r_{12}}{r_{31}} \to 0$, only the first term on the right hand side of Eq.(5b) survives, and

$$\operatorname*{Limit}_{\substack{m_3 \to 0 \\ \alpha < 2}} \langle T_{12}\rangle = \frac{U}{2}\langle r_{12}^{\alpha}\rangle \ . \tag{6b}$$

This is two-thirds of the value for the symmetric case. Since the kinetic energy scales like the inverse of the mean square radius, $\langle p_{ij}^2\rangle \propto 1/\langle r_{ij}^2\rangle$, we see that for $\alpha < 2$ decreasing the mass of particle 3 not only makes the wave function have a larger radius for the motion of particle 3 relative to particles 1 and 2 but also affects the relative motion of 1 and 2 and makes its radius larger.

Similar effects are seen in the hyperfine interaction (5c). For the harmonic oscillator case, $\alpha = 2$, Eq.(5c) simplifies to give

$$[V_{12}^{hyp}]_{\alpha=2} = (3/2)Km_{12}\langle r_{12}\rangle \ . \tag{6c}$$

This is again independent of the motion and mass of particle 3, as expected. However, for $\alpha < 2$ the expression (5c) depends upon the mass of particle 3, and in the limit $m_3 = 0$, the result analogous to (6b) is obtained as all the terms in (5c) vanish except for the first term.

$$\operatorname*{Limit}_{\substack{m_3\to 0\\ \alpha<2}} V_{12}^{hyp} = K\, m_{12}\langle r_{12}^{\alpha-1}\rangle \ . \tag{6d}$$

This result is again two-thirds of the value for the harmonic oscillator case and shows that decreasing the mass of particle 3 increases the overall size of the wave function including the relative motion of particles 1 and 2. However, a simple relation analogous to (6a) does not exist for the hyperfine interaction in the symmetric case because the two terms in the square bracket do not cancel. The kinetic energy has simpler properties than the hyperfine interaction because the virial theorem gives a simple expression for the total kinetic energy and there is no analogous simple expression for the total hyperfine interaction.

Eqs.(6) show another qualitative difference between the harmonic oscillator potential and potentials like the Coulomb and logarithmic potentials which are singular at the origin. Increasing all masses by the same factor decreases the size of the system and increases the hyperfine interaction. This effect can be computed simply by taking the logarithmic derivative of Eq.(5c) and scaling the wave functions

$$\frac{d}{d\ \log\ m}\left[\log V_{12}^{hyp}\right] = \frac{d}{d\ \log\ m}(\log m_{12}) + \frac{d}{d\ \log\ m}\log\langle r_{ij}^{\alpha-1}\rangle \tag{6e}$$

where m denotes a mass scale parameter. Since all masses scale by the same factor, $d(\log m_{ij}) \equiv d(\log m)$ for all ij and

$$\frac{d}{d\ \log\ m}\left[\log V_{12}^{hyp}\right] = 1 + \frac{1-\alpha}{2+\alpha} = \frac{3}{2+\alpha} \tag{6f}$$

There are two independent contributions, the direct contribution from m_{12} and the effect of scaling of the wave function. For the oscillator potential the two factors m_{12} and $\langle r_{12}^{\alpha-1}\rangle$ work in opposite directions, with the increase in m_{12} dominating over the decrease in $\langle r_{12}^{\alpha-1}\rangle$. For potentials with $\alpha < 1$, the two factors in (6d) work in the <u>same</u> direction. The result is a strong α dependence with values of 3/4, 3/2 and 3 respectively for the oscillator, log and Coulomb potentials. For the case where m_3 changes and m_{12} is unchanged, the analog of Eq.(6e) shows that the entire effect comes from changes in $\langle r_{12}^{\alpha-1}\rangle$ which has opposite signs for the same scale change in the wave function in the cases of $\alpha > 1$ and $\alpha < 1$.

Eqs.(5) show that the relative kinetic energy and the hyperfine interaction depend not only upon the size of the system but also on expectation values of operators depending upon correlations between different pairs. We now attempt to obtain a quantitative estimate of the effect which is insensitive to assumptions about correlations. For this reason we work with the relative kinetic energy (5a) which is less sensitive to these correlations and should have the same qualitative scaling behavior as the hyperfine interaction. Note, however, that in any attempt to solve the three body problem numerically, the expression (5c) may give a better value for the hyperfine interaction than the direct calculation of wave functions at the origin, because of arguments demonstrated in Ref.8.

We now attempt to estimate this effect quantitatively. Our unperturbed wave function Ψ_o is defined as the exact solution of the three-body problem with the real two-body interaction for the case where all quarks have equal masses given by Eq.(2). The change in the wave function produced by a small change δm_3 in m_3 is assumed to be expressed by changing the scale of the relative co-ordinate r_{12} by a factor $1 + \kappa$ and the scales of the relative co-ordinates r_{23} and r_{31} by a factor $1 + \lambda$, where κ and λ are small. The expectation value of any function $F(r_{12}, r_{23}, r_{31})$ of the relative co-ordinates in this perturbed wave function is then

$$\langle\Psi|F(r_{12},r_{23},r_{31})|\Psi\rangle = \langle\Psi_o|F[(1+\kappa)r_{12},(1+\lambda)r_{23},(1+\lambda)r_{31}]|\Psi_o\rangle \ . \quad (7)$$

The change in the expectation values of the kinetic energy T_{ij} of the relative motion of the ij pair is given to first order in δm_3, κ and λ by

$$\delta\langle T_{12}\rangle = \frac{\delta(p_{12}^2)}{2m_{12}} = -2\kappa\ \langle T_{12}\rangle \quad (8a)$$

$$\delta\langle T_{31}\rangle = \delta\langle T_{23}\rangle = \delta\left(\frac{\langle p_{23}^2\rangle}{2m_{23}}\right) = -2\lambda\ \langle T_{23}\rangle - \frac{\delta m_{23}}{m_{23}}\langle T_{23}\rangle \ . \quad (8b)$$

The values of $\langle T_{ij}\rangle$ can also be calculated by substituting Eq.(7) into the expression (5a) obtained from the virial theorem and the analogous expression for $\langle T_{23}\rangle$ obtained by cyclic permutation on Eq.(5a). Thus

$$\langle T_{12}\rangle + \delta\langle T_{12}\rangle = \frac{U}{2}\,\langle (1+\kappa)^{\alpha} r_{12}^{\alpha} + \frac{1}{4}\,(1+\lambda)^{\alpha}(r_{23}^{\alpha} + r_{31}^{\alpha})$$

$$+ \frac{1}{4}\,[(1+\kappa)^{2}(1+\lambda)^{\alpha-2} r_{12}^{2}(r_{23}^{\alpha-2} + r_{31}^{\alpha-2}) - (1+\lambda)^{\alpha}(r_{31}^{2} r_{23}^{\alpha-2} + r_{23}^{2} r_{31}^{\alpha-2})\rangle \tag{9a}$$

$$\langle T_{23}\rangle + \delta\langle T_{23}\rangle = \frac{U}{2}\,\langle (1+\lambda)^{\alpha} r_{23}^{\alpha} + \frac{1}{4}\,(1+\lambda)^{\alpha} r_{31}^{\alpha} + (1+\kappa)^{\alpha} r_{12}^{\alpha}$$

$$+ \frac{1}{4}\,[(1+\lambda)^{2} r_{23}^{2}(1+\lambda)^{\alpha-2} r_{31}^{\alpha-2} + (1+\kappa)^{\alpha-2} r_{12}^{\alpha-2} \tag{9b}$$

$$- (1+\kappa)^{2}(1+\lambda)^{\alpha-2} r_{12}^{2} r_{31}^{\alpha-2} - (1+\lambda)^{2}(1+\kappa)^{\alpha-2} r_{31}^{2} r_{12}^{\alpha-2}\rangle\ .$$

Combining Eqs.(6a), (8) and (9) and using the permutation symmetry of the wave function then gives to first order in κ and λ,

$$\delta\langle T_{12}\rangle = \frac{2}{3}\,\langle T_{ij}\rangle\,\left[\alpha\kappa + \frac{1}{2}\,\alpha\lambda + (\kappa - \lambda)\phi_{\alpha}\right] = -2\kappa\langle T_{ij}\rangle \tag{10a}$$

$$\delta\langle T_{23}\rangle = \frac{2}{3}\,\langle T_{ij}\rangle\,\left[\frac{5}{4}\,\alpha\lambda + \frac{1}{4}\,\alpha\kappa + \frac{1}{2}\,(\lambda - \kappa)\phi_{\alpha}\right] = \left(-2\lambda - \frac{\delta m_{23}}{m_{23}}\right)\langle T_{ij}\rangle \tag{10b}$$

where

$$\phi_{\alpha} \equiv \frac{\langle r_{ij}^{2} r_{jk}^{\alpha-2}\rangle}{\langle r_{ij}^{\alpha}\rangle}\,, \quad k \neq i\ . \tag{10c}$$

Solving these equations (10) for κ and λ gives

$$\kappa = -\,\frac{2(2\phi_{\alpha}-\alpha)}{3(\alpha+2)(\alpha+4+2\phi_{\alpha})}\,\frac{\delta m_{23}}{m_{23}} \tag{11a}$$

$$\lambda = -\,\frac{4(\alpha+\phi_{\alpha}+3)}{3(\alpha+2)(\alpha+4+2\phi_{\alpha})}\,\frac{\delta m_{23}}{m_{23}}\ . \tag{11b}$$

From Eq.(12c) we see that $\kappa = 0$ for the harmonic oscillator potential where $\alpha = 2$ and $\phi_\alpha = 1$. This again shows no effect on r_{12} from changing m_{23} in the harmonic case.

For the log potential, $\alpha = 0$

$$\kappa_o = - \frac{\phi_o}{3\phi_o + 6} \left(\frac{\delta m_{23}}{m_{23}}\right) . \tag{12a}$$

If the hyperfine interaction V_{ij}^{hyp} between quarks i and j is assumed to scale like the square of the wave function of the pair ij at the origin,

$$\frac{\delta \langle V_{12}^{hyp} \rangle}{\langle V_{12}^{hyp} \rangle} = \frac{\delta \langle \Psi_{12}(o)^2 \rangle}{\langle \Psi_{12}(o)^2 \rangle} = -3\kappa_o = \left(\frac{\phi_o}{\phi_o + 2}\right)\left(\frac{\delta m_{23}}{m_{23}}\right) . \tag{12b}$$

A better result is presumably obtainable by using Eq.(5c), but is more complicated because it depends on several correlation parameters. We therefore use (12b).

In the unperturbed state, $m_{23} = m_q/2$, where m_q is given by equation (2). For the Σ and Ξ systems which change m_3 in opposite directions from the symmetric state,

$$\left(\frac{\delta m_{23}}{m_{23}}\right)_\Xi = -\left(\frac{\delta m_{23}}{m_{23}}\right)_\Sigma = \frac{(m_s - m_u)}{2(m_s + m_u)} \sim \frac{1}{10} \tag{13}$$

where the numerical value is obtained by setting $m_s/m_u = 3/2$. Then

$$\frac{(\delta V^{hyp})_\Xi - (\delta V^{hyp})_\Sigma}{V^{hyp}} = \left(\frac{\phi_o}{\phi_o + 2}\right) \left[\frac{(m_s - m_u)}{m_s + m_u}\right] \sim \frac{1}{5} \left[\frac{\phi_o}{\phi_o + 2}\right] . \tag{14}$$

Evaluation of ϕ_o requires the knowledge of the wave function. However from the symmetry

$$\phi_o = \langle r_{12}^2 r_{23}^{-2} \rangle = \langle r_{23}^2 r_{12}^{-2} \rangle = 1 + \frac{1}{2} \langle (r_{12} r_{23}^{-1} - r_{23} r_{12}^{-1})^2 \rangle \ > 1 \ . \tag{15}$$

This gives the lower bound

$$\frac{\delta V^{hyp}}{V^{hyp}} > \frac{1}{3} \frac{(m_s - m_u)}{(m_s + m_u)} \sim \frac{1}{15} \ . \tag{16}$$

The effect is thus seen to have the right sign and the right order of magnitude. If harmonic oscillator wave functions are used to evaluate ϕ_o, using the Isgur-Karl relative variables ρ and λ, $\phi_o = 2.5$ is obtained, which gives 1/9 or 11% for $\delta V_{hyp}/V_{hyp}$.

A similar effect should be present in the hyperfine interaction between two nonstrange quarks when the mass of the third quark is changed from m_u to m_s. The prediction[2]

$$M_\Delta - M_N = (1/2)(2M_{\Sigma^*} + M_\Sigma - 3M_\Lambda) \tag{17}$$

relates the ud hyperfine splitting in the Δ-N system to the Σ^*-Σ-Λ system. Here the LHS is 294 MeV and the RHS is 307 MeV, giving a smaller effect of 4% in the same direction as the discrepancy in (1), and attributable to the change in the size of the wave function. The reason for the smallness of the effect in this case, as compared with (1) and (16) is unclear. However, the prediction (17) is not as firm as (1) because of differences in the wave functions and the necessity to subtract out the contribution from strange-nonstrange pairs. The nucleon, Σ and Ξ all have the same spin couplings and the SU(6) breaking in the wave functions due to the hyperfine interaction itself is the same in all cases and should not affect the relation (1). However, the spin couplings in the Λ are different and the hyperfine interaction which gives attraction between the ud pair and repulsion for the us and ds pairs has no simple counterpart in the nucleon and Δ. These effects could give additional contributions to change the deviation from the relation (17).

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